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INEQUALITIES INVOLVING EIGENVALUES AND DIAGONAL ENTRIES OF A NONNEGATIVE MATRIX[1]

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Abstract

In a recent work [1], a family of symmetric polynomials of the eigenvalues of a square complex matrix was defined and a one-to-one relation with a corresponding family of polynomials of matrix entries was established. Several consequences and applications of this result were pointed out and discussed. In this paper, we summarize results which follow from this work and particularly refer to the class of nonnegative matrices.

Key words: nonnegative matrices, symmetric polynomials, eigenvalue inequalities

1. Introduction. Given a matrix $A = [a_{ij}] \in M_n(C)$, the multiset of eigenvalues $\lambda_1, \dots, \lambda_n$ of A (spectrum of A) will be denoted by $\Lambda = (\lambda_1, \dots, \lambda_n)$. We shall write $A \geq 0$ ($A > 0$) if $A = [a_{ij}] \in M_n(R)$ with $a_{ij} \geq 0$ ($a_{ij} > 0$), $i, j = 1, \dots, n$. It is well known that the spectrum of a matrix $A \geq 0$ contains a nonnegative eigenvalue (Perron root of A) which is greater or equal to the absolute value of any other eigenvalue. In the spectral theory of nonnegative matrices there is plenty of results in the form of inequalities which give upper and lower estimates of the Perron root, establish relations among the Perron root and the other eigenvalues, provide criteria for specific matrix properties, etc. Several other types of inequalities are also known to hold for the eigenvalues of a nonnegative matrix. We shall briefly mention some of them.

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If $A \geq 0$ and s_k denotes

$$s_k = \sum_{i=1}^{n} \lambda_i^k, \quad k = 0, 1, 2, \ldots, \tag{1}$$

then we have the power sum inequalities

$$s_k \geq 0, \quad k = 0, 1, \ldots \tag{2}$$

Indeed, the nonnegativity of A implies $\mathrm{tr}(A^k) \geq \sum\limits_{i=1}^{n} a_{ii}^k \geq 0$ and since $s_k = \mathrm{tr}(A^k)$, inequalities (2) follow immediately.

Less obvious relations among the power sums are obtained by Loewy, London and Johnson, e.g. see [2]. In this case, we have

$$n^{m-1} s_{km} \geq s_k^m, \quad k, m = 1, 2, \ldots. \tag{3}$$

Inequalities (3) can be deduced from the well known Hölder's inequality and it is easily seen that the equality in (3) is attained if A is a scalar matrix, i.e. $A = \alpha I$, $\alpha \geq 0$. Also, it follows from (3) that $n^{m-1} s_m \geq s_1^m$, $m = 1, 2, \ldots$ which together with the obvious inequality $s_1 \geq 0$ imply the power sum inequalities (2).

Another type of inequalities follow from the result in [3], where it is shown that the elementary symmetric functions of the eigenvalues of an M-matrix satisfy the classical Newton's inequalities. In particular, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a nonnegative matrix A with Perron root $\lambda_1 = \max\limits_{1 \leq i \leq n} |\lambda_i|$. Then $\lambda_1 I - A$ is an M-matrix with eigenvalues $0, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$ which satisfy the inequalities

$$\frac{e_j^2}{\binom{n}{j}^2} \geq \frac{e_{j-1}}{\binom{n}{j-1}} \cdot \frac{e_{j+1}}{\binom{n}{j+1}}, \quad j = 1, \ldots, n-1, \tag{4}$$

where e_j is the j-th elementary symmetric function of the sequence $0, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$ for $j = 0, 1, \ldots, n$. In the general case, inequalities (3) and (4) are independent. More precisely, it is shown in [3] that neither the two conditions (2) and (3) together imply (4) nor the two conditions (2) and (4) together imply (3). However, a relation between (3) and the inequalities of Newton is pointed out in [4], where it is shown that each inequality in (3) for $m = 2$ and $k = 1, 2, \ldots$ follows from the first Newton's inequality applied to the eigenvalues of A^k for $k = 1, 2, \ldots$, respectively.

Apart from their independent interest, inequalities involving eigenvalues of a nonnegative matrix play an important role in the study of certain properties of other matrix classes and matrix-theoretic problems. Typical examples in this respect provide the class of M-matrices [5,6] and the various forms of the inverse eigenvalue problem for nonnegative matrices [7]. In the latter case, it is clear that each of the inequalities (2)–(4) provides a necessary condition for an n-tuple of complex numbers $\lambda_1, \dots, \lambda_n$ to be the spectrum of a nonnegative matrix.

In this paper, we present a family of inequalities relating the eigenvalues and diagonal entries of a nonnegative matrix. These inequalities follow from the main result in [1] and can be viewed as a broad generalization of the power sum inequalities (2). In fact, (2) turns out to be the simplest special case in our family of inequalities. Another set of inequalities satisfied by the power sums of the eigenvalues of a nonnegative matrix is also given.

2. Main result. The following index sets will be used in the sequel. Given a positive integer n, let N be the set $N = \{1, 2, \dots, n\}$. For integers m and n with $1 \le m \le n$, we shall denote by $Q_{m,n}$ the set of all sequences of the form $i = (i_1, \dots, i_m)$ such that $i_1, \dots, i_m \in N$ and $i_1 < \dots < i_m$. It can be easily seen that $Q_{m,n}$ has $\binom{n}{m}$ elements. We shall also use sequences of nonnegative integers of the form $j = (j_1, \dots, j_m)$ and in this case $|j|$ will denote the sum $|j| = j_1 + \dots + j_m$. The positions of elements in $j = (j_1, \dots, j_m)$ will matter so that the number of all sequences satisfying $|j| = k$ for some nonnegative integer k, is $\binom{k+m-1}{k}$. With this notation, the complete homogeneous symmetric polynomial of degree k in m independent variables $x_1, \dots, x_m$ is defined as

$$h_k(x_1, \dots, x_m) = \sum_{|j|=k} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}. \tag{5}$$

For a matrix $A \in M_n(C)$, the principal submatrix obtained by deleting rows and columns of A with indexes $i_1, \dots, i_m$ is denoted by $A(i_1, \dots, i_m)$. A special notation is used for the diagonal entries of a matrix, i.e., $[A]_p$ denotes the diagonal element of A at position (p, p). This notation is particularly suitable in identifying diagonal elements of powers of a given submatrix of A. Thus, $[A(i_1, \dots, i_m)^q]_p$ denotes the diagonal element at position (p, p) of the q-th power of $A(i_1, \dots, i_m)$; note that, in this case, p ranges from 1 to $n - m$.

Now, let $A = [a_{ij}] \in M_n(C)$ be given with spectrum $\Lambda = (\lambda_1, \dots, \lambda_n)$. The following family of polynomials of $\lambda_1, \dots, \lambda_n$ is defined and studied in [1]. For each m, $1 \le m \le n$, let $s_{k,m}(\Lambda)$ be the polynomial given by

$$s_{k,m}(\Lambda) = \sum_{i \in Q_{m,n}} h_k(\lambda_{i_1}, \dots, \lambda_{i_m}), \quad k = 0, 1, \dots \tag{6}$$

Note that the special cases $m = 1$ and $m = n$ reduce to $s_{k,1}(\Lambda) = s_k$ and $s_{k,n}(\Lambda) = h_k(\lambda_1, \dots, \lambda_n)$ for $k = 0, 1, \dots$. Another family of polynomials of the entries of A is also defined as follows. Given m, $1 \le m \le n$, let $p_{k,m}(A)$ be the polynomial of $a_{i,j}$, $i,j=1,\ldots,n$ given by

$$(7) \qquad p_{k,m}(A) = \sum_{i\in Q_{m,n}} \sum_{|j|=k} [A^{j_1}]_{i_1} [A(i_1)^{j_2}]_{i_2-1} \cdots [A(i_1,\ldots,i_{m-1})^{j_m}]_{i_m-m+1}$$

for each $k=0,1,\ldots$. We note that $s_{k,m}(\Lambda)$ is a homogeneous symmetric polynomial of degree k, $p_{k,m}(A)$ is a homogeneous polynomial of the same degree and both polynomials have positive integer coefficients. The main result obtained in [1] proves the following relation between the two families of polynomials (6) and (7).

Theorem 1. *Let $A \in M_n(C)$ and $\Lambda = (\lambda_1,\ldots,\lambda_n)$ be the spectrum of A. For each m, $1 \le m \le n$,*

$$(8) \qquad s_{k,m}(\Lambda) = p_{k,m}(A), \quad k=0,1,\ldots.$$

The proof of Theorem 1 utilizes analytical tools involving scalar and matrix power series expansions. Equality (8) implies some important properties of polynomials $p_{k,m}(A)$, $1 \le m \le n$ and $k=0,1,\ldots$. Since the spectrum Λ of a square matrix is invariant under a similarity transformation, it follows from (8), that $p_{k,m}(A)$ is also invariant under a similarity transformation of A, i.e.,

$$p_{k,m}(A) = p_{k,m}(S^{-1}AS)$$

for any nonsingular matrix S. Since AB and BA have the same spectrum for any $A, B \in M_n(C)$, it also follows that

$$p_{k,m}(AB) = p_{k,m}(BA).$$

From the definition of $p_{k,m}(A)$ given by (7), it can be easily seen that if A is a nonnegative matrix then $p_{k,m}(A)$ takes on nonnegative values for any $1 \le m \le n$ and $k=0,1,\ldots$. This fact together with the equality (8) in Theorem 1 enables us to derive a family of inequalities involving eigenvalues and diagonal entries of a nonnegative matrix. The result is formulated in the next theorem.

Theorem 2. *Let $A=[a_{ij}] \in M_n(R)$, $\Lambda = (\lambda_1,\ldots,\lambda_n)$ and $A \ge 0$. For each $1 \le m \le n$ and $k=0,1,\ldots$, the following inequality holds*

$$(9) \qquad \sum_{i\in Q_{m,n}} h_k(\lambda_{i_1},\ldots,\lambda_{i_m}) \ge \sum_{i\in Q_{m,n}} h_k(a_{i_1 i_1},\ldots,a_{i_m i_m}).$$

Proof. Let $\mathrm{diag}(A)$ denote the diagonal matrix with diagonal entries $a_{11},\ldots,$ a_{nn}. Since A is a nonnegative matrix, it is easily seen that

$$(10) \qquad [A^k]_i \ge a_{ii}^k, \quad i=1,\ldots,n, \; k=0,1,\ldots$$

We recall that $[A^k]_i$ in inequality (10) denotes the diagonal element at position (i,i) of the k-th power of A. By applying this inequality to the right-hand side of (7), it is obtained

$$p_{k,m}(A) \geq p_{k,m}(\mathrm{diag}(A)) = \sum_{i \in Q_{m,n}} h_k(a_{i_1 i_1}, \ldots, a_{i_m i_m}). \tag{11}$$

Now, inequality (8) follows by taking into account the definition of $s_{k,m}(\Lambda)$ in (6), equality (8) in Theorem 1 and inequality (11).

Two special cases of Theorem 2 can be pointed out. In the simplest special case $m = 1$, the following well known inequalities are obtained from (9):

$$\sum_{i=1}^{n} \lambda_i^k \geq \sum_{i=1}^{n} a_{ii}^k, \quad k = 0, 1, 2, \ldots . \tag{12}$$

Since $s_k = \sum_{i=1}^{n} \lambda_i^k$ and $a_{ii} \geq 0$, $i = 1, \ldots, n$, (12) obviously implies the power sum inequalities (2). The other special case provides a relation between the complete homogeneous symmetric polynomials of the eigenvalues and diagonal entries of a nonnegative matrix. In particular, the case $m = n$ in (9) yields the following inequalities:

$$h_k(\lambda_1, \ldots, \lambda_n) \geq h_k(a_{11}, \ldots, a_{nn}), \quad k = 0, 1, \ldots . \tag{13}$$

By considering $\lambda_1, \ldots, \lambda_n$ as independent variables, polynomials $s_{k,m}(\Lambda)$ defined by (6) can be viewed as elements of the ring $\Lambda_{n,Z}$ of all symmetric polynomials of n variables with integer coefficients. It is well known that the power sums $s_k = \sum_{i=1}^{n} \lambda_i^k$, $k = 0, 1, \ldots$, provide a basis in $\Lambda_{n,Z}$, so that any element in the ring $\Lambda_{n,Z}$ can be uniquely written as a polynomial of the power sums of $\lambda_1, \ldots, \lambda_n$. However, finding the explicit form in which a polynomial of $\lambda_1, \ldots, \lambda_n$ in $\Lambda_{n,Z}$ is written as a linear combination of the elements of a given basis is not always straightforward and the study of relationships among the different bases in $\Lambda_{n,Z}$ is an important subject in the theory of symmetric functions [8]. In what follows, we consider the case $m = 2$ in (6) and state an explicit formula for $s_{k,2}(\Lambda)$ as a polynomial of the power sums $s_0, \ldots, s_k$ for $k = 0, 1, \ldots$. This result is formulated in the next proposition.

Proposition 1. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$, $s_{k,2}(\Lambda)$ be given by (6) with $m = 2$ and s_k denote the power sums (1). Then*

$$s_{k,2}(\Lambda) = \frac{1}{2} \left(\sum_{p+q=k} s_p s_q - (k+1) s_k \right), \quad k = 0, 1, \ldots \tag{14}$$

Proof. By expanding the right-hand side of (14), it is obtained

$$(15)\quad \sum_{p+q=k} s_p s_q - (k+1)s_k$$
$$= \sum_{p+q=k} (\lambda_1^p + \cdots + \lambda_n^p)(\lambda_1^q + \cdots + \lambda_n^q) - (k+1)(\lambda_1^k + \cdots + \lambda_n^k)$$
$$= \sum_{i_1,i_2=1}^{n} \sum_{p+q=k} \lambda_{i_1}^p \lambda_{i_2}^q - (k+1)(\lambda_1^k + \cdots + \lambda_n^k)$$
$$= 2 \sum_{1\leq i_1<i_2\leq n} \sum_{p+q=k} \lambda_{i_1}^p \lambda_{i_2}^q = 2 \sum_{i\in Q_{2,n}} h_k(\lambda_{i_1}, \lambda_{i_2}).$$

Now, (14) follows immediately from equalities (15) and the definition of $s_{k,2}(\Lambda)$ in (6).

The above proposition is taken from [1] and the proof in this reference is based on the expansion of two power series and comparison of their coefficients of the equal powers. Obviously, the proof given here is shorter and straightforward. Equality (14) enables us to obtain the following result.

Theorem 3. *Let $A = [a_{ij}] \in M_n(R)$, $\Lambda = (\lambda_1, \dots, \lambda_n)$ and $A \geq 0$. For each $k = 0, 1, \dots,$ the following inequality holds*

$$(16)\qquad \frac{1}{2}\left(\sum_{p+q=k} s_p s_q - (k+1)s_k\right) \geq \sum_{i\in Q_{2,n}} h_k(a_{i_1 i_1}, a_{i_2 i_2}).$$

It is easily seen that (17) follows from inequality (9) with $m = 2$, the definition of $s_{k,2}(\Lambda)$ in (6) and equality (14). Thus, in the special case $m = 2$, the above theorem gives an equivalent form of (9) expressed in terms of the power sums $s_0, \dots, s_k$. Since A is a nonnegative matrix, the right-hand side of (17) is nonnegative and Theorem 3 yields a family of power sum inequalities in the form

$$\left(\sum_{p+q=k} s_p s_q - (k+1)s_k\right) \geq 0, \quad k = 0, 1, \dots.$$

3. Conclusion. The main result of the paper is formulated in Theorem 2 and it represents a family of inequalities involving symmetric polynomials of the eigenvalues and diagonal entries of a nonnegative matrix. This result reveals specific properties of the class of nonnegative matrices and it also provides a generalization of some well known and widely used inequalities. In Proposition 1, we have obtained an explicit formula for $s_{k,2}(\Lambda)$ as a polynomial of the power sums $s_0, \dots, s_k$ and we have used this result in Theorem 3 in order to give an equivalent form of inequalities (9) in the case $m = 2$. An interesting question arising from the result in Theorem 2 is the following. It would be useful to find a complete characterization of the class of all symmetric polynomials in n variables which satisfy inequalities (9) for a nonnegative matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and diagonal entries $a_{11}, \dots, a_{nn}$. Certainly, the power sums (1) and the complete homogeneous symmetric polynomials (5) belong to this class.

Some areas of possible application of the results can be pointed out as follows. The theory of nonnegative matrices [2] is an area of continuous interest and active research and here, our eigenvalue inequalities may be useful in characterizing certain properties of this important matrix class as well as in studying some other related classes of matrices such as M-matrices. The obtained results may also be relevant to some variants of the inverse eigenvalue problem for nonnegative matrices [7]. In particular, it is clear that inequalities (9) give a necessary condition for an n-tuple of complex numbers $\lambda_1, \dots, \lambda_n$ to be the spectrum of a nonnegative matrix with diagonal entries $a_{11}, \dots, a_{nn}$. Another area of application is the theory of positive dynamical systems [9, 10] where a linear homogeneous discrete-time system is described by the state space representation

$$x(k+1) = Ax(k), \tag{17}$$

where x is the n-dimensional vector of state variables, $A \in M_n(R)$ is the state matrix and k denotes the discrete time. System (17) is said to be positive if for any positive initial state vector the trajectory of the system remains positive for all values of k. An immediate consequence of this definition is that (17) is a positive system if and only if the state matrix A has positive elements. In this area of systems research, theoretical results characterizing the properties of a positive matrix A, and especially the properties of its eigenvalues, provide an important tool in the analysis and control of this type of systems.

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